Jiří Fišer*

Department of Mathematical Analysis and Applications of Mathematics, Natural Science Faculty of Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic

Received 20th November 1997

KEY WORDS: TRIPLE LOGISTIC MAP, FRACTAL DIMENSION.

ABSTRACT: Structure of attractors of a triple logistic map is treated in terms of fractal dimension. This investigation continues the one in [2], [3], where the detection of attractors and their basins of attraction have been studied.

Experimental determination of a dimension value from data for an experimental dynamical processes can provide information on the dimensionality of the phase space required of a mathematical dynamical system used to model the observations (see e.g. [7] and the references therein). These models can describe many important nonlinear phenomena in physics, biology, etc.

Multiple logistic maps exhibit a complex of chaotic effects, when bifurcations occur (see e.g. [1], [4] for double logistic maps and [2], [3] for a triple one). Using the critical manifold technique or numerical explorations, one can visualize the associated pairs “attractor/basin”. In order to have a more concrete information concerning their structure, we can calculate several characteristics like fractal dimension, entropy, correlations, or so. Here, the first one will be taken into account.

By a triple logistic map we mean

\[
T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (1 - \lambda)x + 4\lambda y(1 - y) \\ (1 - \lambda)y + 4\lambda z(1 - z) \\ (1 - \lambda)z + 4\lambda x(1 - x) \end{pmatrix}, \quad \lambda \in [0, 1].
\]

* Supported by grant no. 311-03-001 of Palacký University, Olomouc.
In [2], [3], we have already discovered some properties of this mapping, namely [3]:

- $T$ has eight fixed points, which are independent on parameter $\lambda$;
- No fixed point of $T$ is stable;
- Some attractors of $T$ are chaotic (see Fig. 1);

![Figure 1: Examples of attractors for the triple logistic map](image)

- Nevertheless, their basins of attraction seem to be generically “non-chaotic” (see Fig. 2);

![Figure 2: Sections of basins of attraction for some values of parameter $\lambda$.](image)
Figure 3: Visualization of basin of attraction for $\lambda = 0.2$.

A visualization of a “3D” basin of attraction can be done by a “manual” composition of numerically obtained sections of that basin (we used “DYNAMICS” software package (see [6]) and CorelDraw, see Fig. 3);

Figure 4: Method of critical surfaces for $\lambda = 0.9$.

The application of a critical manifold method (to detect absorbing sets, (see [5])) for non-trivial cases was not so effective, we need powerful computer algebra system (MAPLE), particularly “implicitplot” procedure (see Fig. 4);

Hence, our main goal will be related, as already pointed out, to the estimates of fractal dimension.
The box-counting dimension (for the definition see below) of some sets can be non-integer. Such sets are called fractals, while, in the context of dynamics, attracting sets with fractal properties are called strange attractors.

We count the number of \( \varepsilon \)-cubes \( \widetilde{N}(\varepsilon) \) needed to cover the set.

The box-counting dimension is defined by

\[
D_0 = \lim_{\varepsilon \to 0} \frac{\ln \widetilde{N}(\varepsilon)}{\ln \left(\frac{1}{\varepsilon}\right)},
\]

where

\[
D_0 = \begin{cases} 
0 & \text{for a set of a finite number of points,} \\
1 & \text{for a simple smooth curve,} \\
\text{etc.} & 
\end{cases}
\]

The box-counting dimension gives the scaling of the number of cubes needed to cover the attractor, but typical orbits can spend most of their time in a small minority of those cubes that are needed to cover the attractor. Some cubes are more important, and so we introduce a natural measure of the attractor.

The natural measure of the typical cube \( C_i \) is

\[
\mu_i = \lim_{T \to \infty} \frac{\eta(C_i, T)}{T},
\]

where \( \eta(C_i, T) \) is the amount of time which the orbit spends in \( C_i \) in the time interval \( 0 \leq t \leq T \).

To take into account the different natural measures of the cubes it is possible to introduce another definition of dimension which generalizes the box-counting dimension. This definition of dimension was formulated in the context of chaotic dynamics by Grassberger (1983) and Hentschel and Procaccia (1983) (for more details see [7]). These authors define a dimension \( D_q \) which depends on a continuous index \( q \),

\[
D_q = \frac{1}{1 - q} \lim_{\varepsilon \to 0} \frac{\ln I(q, \varepsilon)}{\ln \left(\frac{1}{\varepsilon}\right)},
\]

where

\[
I(q, \varepsilon) = \sum_{i=1}^{\widetilde{N}(\varepsilon)} \mu_i^q,
\]

and the sum is over all the \( \widetilde{N}(\varepsilon) \) cubes in a grid of unit size \( \varepsilon \) needed to cover the attractor. For \( q > 0 \) cubes with larger \( \mu_i \) have a greater influence in determining the value of \( D_q \). Note that for \( q = 0 \) we have \( I(0, \varepsilon) = \widetilde{N}(\varepsilon) \), and we recover the box-counting dimension definition.
In general, it can be shown that

\[ D_{q_1} \leq D_{q_2} \text{ if } q_1 > q_2. \]

Defining \( D_1 \) by

\[ D_1 = \lim_{\varepsilon \to 0} \sum_{i=1}^{\bar{N}(\varepsilon)} \frac{\mu_i \ln \mu_i}{\ln \varepsilon}, \]

we have

\[ D_1 = \lim_{\varepsilon \to 0} \frac{H_q}{\ln(\varepsilon)}. \]

The quantity \( D_1 \) is called the information dimension.

For convenience, we use the following notation:

\[ D_q = \lim_{\varepsilon \to 0} \frac{H_q}{\ln(\varepsilon)} \]

where

\[ H_0 = \ln \bar{N}(\varepsilon), \quad H_{\frac{1}{2}} = 2 \ln I(\frac{1}{2}, \varepsilon), \quad H_1 = -\sum_{i=1}^{\bar{N}(\varepsilon)} \mu_i \ln \mu_i, \quad H_2 = -\ln I(2, \varepsilon). \]

To determine \( D_q \) one typically generates a long orbit of length \( T \) on the attractor and examines the fraction of time the orbit spends in cubes of \( \varepsilon \)-grid. This gives an approximation to \( \mu_i \) for each cube from which an approximation \( I_T(q, \varepsilon) \) to \( I(q, \varepsilon) \) is obtained. An approximation to the dimensions \( D_q \) can then be obtained by plotting \( H_q \) (with \( I_T(q, \varepsilon) \) instead of \( I(q, \varepsilon) \)) versus \( \ln(\frac{1}{\varepsilon}) \). One can usually fit a straight line to these points and determine \( D_q \)—the slope of the line. The range of \( \varepsilon \) over which such a fitting can be meaningful is limited by the requirement that \( \varepsilon \) must be sufficiently small compared to the attractor size and at small \( \varepsilon \) by statistical fluctuations in determining the \( \mu_i \) (due to necessarily finite amount of data).
\[ \varepsilon = 2^{-k} \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \tilde{N}(\varepsilon) )</th>
<th>( T )</th>
<th>( \varepsilon = 2^{-k} )</th>
<th>( \ln(\frac{1}{\varepsilon}) )</th>
<th>( \ln \tilde{N}(\varepsilon) )</th>
<th>( H_{\frac{1}{2}} )</th>
<th>( H_1 )</th>
<th>( H_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>( 1 \cdot 10^4 )</td>
<td>1/2</td>
<td>0.69</td>
<td>1.61</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>26</td>
<td>( 1 \cdot 10^4 )</td>
<td>1/4</td>
<td>1.39</td>
<td>3.26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>118</td>
<td>( 1 \cdot 10^4 )</td>
<td>1/8</td>
<td>2.08</td>
<td>4.77</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>415</td>
<td>( 1 \cdot 10^5 )</td>
<td>1/16</td>
<td>2.77</td>
<td>6.03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1539</td>
<td>( 1 \cdot 10^5 )</td>
<td>1/32</td>
<td>3.47</td>
<td>7.34</td>
<td>7.07</td>
<td>6.86</td>
<td>6.54</td>
</tr>
<tr>
<td>6</td>
<td>6132</td>
<td>( 2 \cdot 10^5 )</td>
<td>1/64</td>
<td>4.16</td>
<td>8.72</td>
<td>8.42</td>
<td>8.16</td>
<td>6.54</td>
</tr>
<tr>
<td>7</td>
<td>23922</td>
<td>( 3 \cdot 10^5 )</td>
<td>1/128</td>
<td>4.85</td>
<td>10.08</td>
<td>9.77</td>
<td>9.48</td>
<td>8.92</td>
</tr>
<tr>
<td>8</td>
<td>86743</td>
<td>( 5 \cdot 10^5 )</td>
<td>1/256</td>
<td>5.55</td>
<td>11.37</td>
<td>11.09</td>
<td>10.79</td>
<td>10.07</td>
</tr>
</tbody>
</table>

Figure 5: Determination of the dimensions (the slopes)
<table>
<thead>
<tr>
<th>$k$</th>
<th>$\tilde{N}(\varepsilon)$</th>
<th>$T$</th>
<th>$\varepsilon = 2^{-k}$</th>
<th>$\ln(\frac{1}{\varepsilon})$</th>
<th>$\ln \tilde{N}(\varepsilon)$</th>
<th>$H_{\frac{1}{2}}$</th>
<th>$H_1$</th>
<th>$H_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>$5 \cdot 10^3$</td>
<td>$1/2$</td>
<td>0.69</td>
<td>1.61</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>17</td>
<td>$5 \cdot 10^3$</td>
<td>$1/4$</td>
<td>1.39</td>
<td>2.83</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>53</td>
<td>$5 \cdot 10^3$</td>
<td>$1/8$</td>
<td>2.08</td>
<td>3.97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>118</td>
<td>$1 \cdot 10^4$</td>
<td>$1/16$</td>
<td>2.77</td>
<td>4.77</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>301</td>
<td>$2 \cdot 10^4$</td>
<td>$1/32$</td>
<td>3.47</td>
<td>5.71</td>
<td>5.22</td>
<td>4.96</td>
<td>4.70</td>
</tr>
<tr>
<td>6</td>
<td>739</td>
<td>$4 \cdot 10^4$</td>
<td>$1/64$</td>
<td>4.16</td>
<td>6.61</td>
<td>5.95</td>
<td>5.64</td>
<td>5.27</td>
</tr>
<tr>
<td>7</td>
<td>1600</td>
<td>$6 \cdot 10^4$</td>
<td>$1/128$</td>
<td>4.85</td>
<td>7.38</td>
<td>6.72</td>
<td>6.35</td>
<td>5.94</td>
</tr>
<tr>
<td>8</td>
<td>3267</td>
<td>$8 \cdot 10^4$</td>
<td>$1/256$</td>
<td>5.55</td>
<td>8.09</td>
<td>7.45</td>
<td>7.05</td>
<td>6.57</td>
</tr>
</tbody>
</table>

Figure 6: The dimensions are around 1

\[ \lambda = 0.48704 \]

\[ y=1.10x+2.00 \quad (D_0=1.10) \]
\[ y=1.05x+1.60 \quad (D_{1/2}=1.05) \]
\[ y=1.00x+1.50 \quad (D_1=1.00) \]
\[ y=0.95x+1.30 \quad (D_2=0.95) \]
<table>
<thead>
<tr>
<th>$k$</th>
<th>$\tilde{N}(\varepsilon)$</th>
<th>$T$</th>
<th>$\varepsilon = 2^{-k}$</th>
<th>$\ln(1/\varepsilon)$</th>
<th>$\ln \tilde{N}(\varepsilon)$</th>
<th>$H_{\frac{1}{2}}$</th>
<th>$H_1$</th>
<th>$H_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>$1 \cdot 10^4$</td>
<td>$1/2$</td>
<td>0.69</td>
<td>2.08</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>36</td>
<td>$1 \cdot 10^4$</td>
<td>$1/4$</td>
<td>1.39</td>
<td>3.58</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>95</td>
<td>$1 \cdot 10^4$</td>
<td>$1/8$</td>
<td>2.08</td>
<td>4.55</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>126</td>
<td>$2 \cdot 10^4$</td>
<td>$1/16$</td>
<td>2.77</td>
<td>4.84</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>192</td>
<td>$2 \cdot 10^4$</td>
<td>$1/32$</td>
<td>3.47</td>
<td>5.26</td>
<td>4.04</td>
<td>3.46</td>
<td>2.87</td>
</tr>
<tr>
<td>6</td>
<td>378</td>
<td>$4 \cdot 10^4$</td>
<td>$1/64$</td>
<td>4.16</td>
<td>5.93</td>
<td>5.22</td>
<td>4.83</td>
<td>4.30</td>
</tr>
<tr>
<td>7</td>
<td>952</td>
<td>$6 \cdot 10^4$</td>
<td>$1/128$</td>
<td>4.85</td>
<td>6.86</td>
<td>6.32</td>
<td>5.92</td>
<td>5.32</td>
</tr>
<tr>
<td>8</td>
<td>3055</td>
<td>$8 \cdot 10^4$</td>
<td>$1/256$</td>
<td>5.55</td>
<td>8.02</td>
<td>7.59</td>
<td>7.18</td>
<td>6.55</td>
</tr>
</tbody>
</table>

Figure 7: The dimensions seem to be constant

$\lambda=0.8363$
The fractal dimensions of attractors of a triple logistic map could be determined, provided the limits in the definitions of $D_0$ exist. If so, we can still have problems with a sufficiently
accurate and fast computation. Thus, our investigation can be only regarded as more or less heuristic.

**FRACTAL DIMENSION OF ATTRACTIONS OF A TRIPLE LOGISTIC MAP**

The fractal dimensions are estimated for some attractors of a triple logistic map, namely when the parameter $\lambda$ takes the values $\lambda = 0.48704, 0.599, 0.8363$ and $0.8376$.

In Fig. 5 and Fig. 6, there are examples of standard behaviour of fractal dimensions, i.e. their values are decreasing with increasing $q$. In Fig. 7, one can observe almost constant behaviour. Sometimes, however, obstructions may occur (see Fig. 8). These might be due to the insufficient number of iterates, numerical inaccuracy, or so.

**FRAKTÁLNÍ DIMENZE ATRAKTORŮ TROJITÉHO LOGISTICKÉHO ZOBRAZENÍ**

Pro jisté hodnoty parametru $\lambda (\lambda = 0, 48704, 0, 599, 0, 8363$ a $0, 8376)$ jsou odhadnuty fraktální dimenze atraktorů trojitého logistického zobrazení.

Na obr. 5 a 6 jsou příklady standardního chování fraktálních dimenzí, tj. jejich hodnoty klesají s rostoucím $q$. Na obr. 7 mají téměř konstantní charakter. Mohou se však také vyskytnout komplikace (viz obr. 8) způsobené nedostatečným počtem iterací, numerickými nepřesnostmi, atd.